Review: Lots of integrals!

- double integral

main problem: parametrization of region

→ change of order of integration can simplify problem

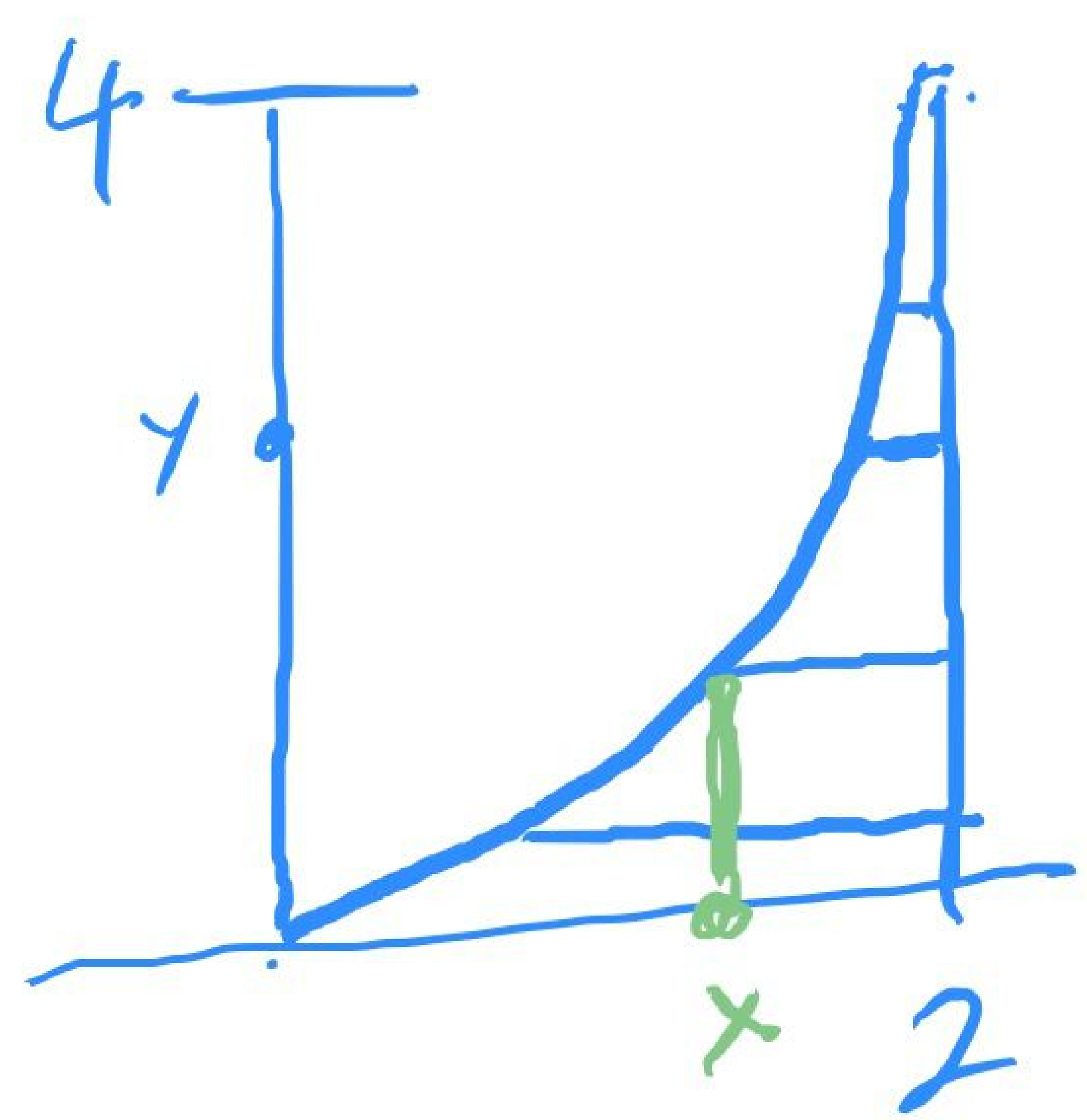
- change of variable formula

example: practice final problem!

calculate $\int_{y=0}^{4} \int_{x=\sqrt{y}}^{2} \cos(x^3)\, dx\, dy$

$x = \sqrt{y}$

$\Rightarrow x^2 = y$



change order of integration

$$\boxed{\begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le x^2 \end{array}}$$

integral $= \int_0^2 \int_0^{x^2} \cos(x^3)\, dy\, dx$

$$= \int_0^2 \cos(x^3)\, y \Big|_0^{x^2} dx$$

$$= \int_0^2 \cos(x^3)\, x^2\, dx$$

$$= \frac{1}{3} \sin(x^3) \Big|_0^2 = \boxed{\frac{1}{3} \sin 8}$$

Change of variable formula (for double and triple integrals)

For double integral:

assume $T^*: D^* \longrightarrow D$ is 1-1 and onto

u, v coord. $\quad$ x, y coordinates

$$\iint_D f(x,y)\,dx\,dy = \iint f(T^*(u,v)) \cdot |dT^*|\,du\,dv$$

where $|dT^*| = \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|$ $\leftarrow$ absolute value of determinant.

Examples: ⓐ polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

i.e. $T^*(r,\theta) = (r\cos\theta, r\sin\theta)$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$ ....

Check: $|dT^*| = r$

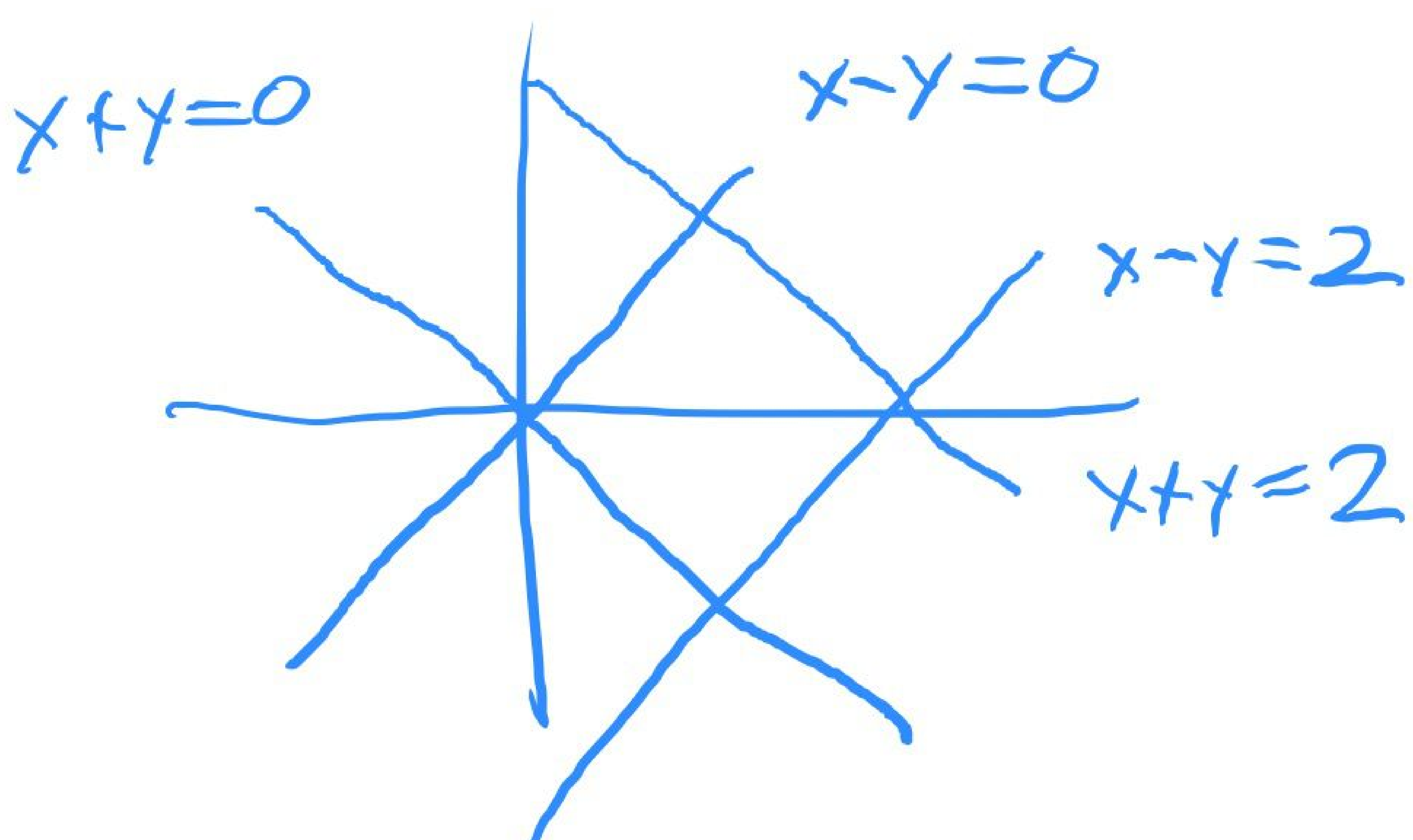
$\Rightarrow$ get integration formula for polar coordinates

$$dx\,dy = r\,dr\,d\theta$$

Problem 2 of practice exam:

Calculate $\iint_D (x+y)e^{x^2-y^2}\,dx\,dy$

over the region D bounded by $\boxed{\begin{array}{ll} x+y=2 & x-y=2 \\ x+y=0 & x-y=0 \end{array}}$



use variables $u = x+y \quad v = x-y$

$\Rightarrow D^*$ given by $\boxed{\begin{array}{l} 0 \leq u \leq 2 \\ 0 \leq v \leq 2 \end{array}}$

Need to express x and y in terms of u and v

$$\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\} \begin{aligned} u+v &= 2x \\ u-v &= 2y \end{aligned} \Rightarrow \boxed{\begin{aligned} x &= \tfrac{1}{2}(u+v) \\ y &= \tfrac{1}{2}(u-v) \end{aligned}}$$

$$\frac{\partial x}{\partial u} = \frac{1}{2} = \frac{\partial x}{\partial v}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2} = -\frac{\partial y}{\partial v}$$

$$\Rightarrow |dT^*| = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = |-1/4 - 1/4|$$

$$\boxed{dT^* = \frac{1}{2}}$$

$$\iint_D \underset{=u}{\underset{\uparrow}{(x+y)}} e^{x^2-y^2} dx\,dy = \int_0^2 \int_0^2 u e^{uv} \cdot \frac{1}{2}\, dv\,du = \frac{1}{2}\int_0^2 e^{uv}\Big|_0^2 du$$

$$x^2-y^2 = (x+y)(x-y) \rightarrow uv$$

$$= \frac{1}{2}\int_0^2 e^{2u} - 1\, du = e^{2u} - \frac{1}{2u}\Big|_0^2$$

$= e^4 - e^0 - (1-0) = \boxed{e^4 - 2}$

Change of variable formula for spherical coordinates

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

i.e. $T(\rho, \phi, \theta) = (x, y, z)$

Can use: $|dT| = \rho^2 \sin\phi$

For linear transformations T:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}$$ A a 3×3 matrix

$\Rightarrow |dT| = |A| =$ absolute value of det A.

Path integral = integral of a function f over a curve C

if curve parametrized by

$$c : [a,b] \to \mathbb{R}^n, \quad n = 2, 3$$

$$\Rightarrow \int_C f \, ds = \int_a^b f(c(t)) \, \|c'(t)\| \, dt$$

Line integral = integral of a vector field F over curve C

$$\boxed{\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) \, dt}$$

- depends on orientation of curve
- otherwise independent of choice of parametrization of curve.

special case if $F = \nabla f$ is a gradient field

$$\Rightarrow \int_C F \cdot ds = \underset{\substack{\uparrow \\ \text{end point} \\ \text{of } C}}{f(c(b))} - \underset{\substack{\uparrow \\ \text{initial} \\ \text{point}}}{f(c(a))}$$

how to detect gradient fields?

3-dim case:

$$\boxed{F \text{ gradient field} \Leftrightarrow \text{curl } F = 0}$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

where $F = (\underset{\substack{\uparrow \\ x\text{-coord.}}}{F_1}, F_2, \underset{\substack{\uparrow \\ z\text{-coord.}}}{F_3})$

2-dim case: $F(x,y) = (P(x,y), Q(x,y))$

F gradient field $\iff$ $\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{= \text{ scalar curl}} = 0$

how to calculate f ?

- guess and check that $F = \nabla f$
- $f(x,y,z) = \int_C F \cdot ds$

  for any curve C: $(0,0,0)$ to (x,y,z)

(could try $c(t) = (tx, ty, tz)$ straight line

or $C = C_1 \cup C_2 \cup C_3$ $\quad$ $C_1(t) = (t,0,0)$ $\quad 0 \le t \le x$
$C_2(t) = (x,t,0)$ $\quad 0 \le t \le y$
$C_3(t) = (x,y,t)$ $\quad 0 \le t \le z$)

Ex. problem in practice exam.

$$F(x,y,z) = \left(-2\sin(2x)e^{5yz},\ 5z\cos 2x\, e^{5yz},\ 5y\cos 2x\, e^{5yz}\right)$$

check: $F = \nabla f$

$$\boxed{f(x,y,z) = \cos 2x\, e^{5yz}}$$

sol. for part (b):

$$\int F \cdot ds = f\left(\frac{\pi}{2}, 1, 1\right) - f(0,0,0)$$